Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 1, pp. 13 – 26

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.1.13

CERTAIN INTEGRAL AND DIFFERENTIAL EQUATIONS INVOLVING HYBRID POLYNOMIALS VIA FACTORIZATION METHOD

RAZIYA SABRI, MOHAMMAD SHADAB, TAEKYUN KIM*, JOSE LUIS LOPEZ-BONILLA AND BYUNG MOON KIM*

Abstract. This article has a motive to derive new classes of differential equations and associated integral equations for some hybrid families of truncated exponential-based Appell polynomials. We derive the recurrence relation, differential equation, integro-differential equation, and integral equation of the truncated exponential based Appell polynomials by using the factorization method. Finally, we also give some illustrative examples.

1. Introduction and preliminaries

The class of the Appell polynomial sequence is one of the significant classes of polynomial sequences [1]. In applied mathematics, theoretical physics, approximation theory, and several other mathematical branches. The set of Appell polynomial sequence is closed under the operation of umbral composition of polynomial sequences. The Appell polynomial sequence can be given by the following generating function

$$A(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1.1}$$

The power series $A(t)$ is given by

$$A(t) = A_0 + \frac{t}{1!} + A_1\frac{t^2}{2!} + A_2 + ... + \frac{t^n}{n!} + ... = \sum_{n=0}^{\infty} A_n\frac{t^n}{n!}, \quad (A_0 \neq 0), \tag{1.2}$$

where $A_i\{i = 1, 2, 3, ...\}$ are real coefficients. It is easy to see that for any $A(t)$, the derivative of $A(t)$ satisfies

$$A_n^{'}(x) = nA_{n-1}(x). \tag{1.3}$$

Some related numbers are shown in table 1.

2010 *Mathematics Subject Classification.* 44A45, 33C45, 05A15, 33E20.

Key words and phrases. Truncated exponential based Appell polynomials; Factorization methods; Differntial equation; Integral equations, Integrodifferential equation.

*Corresponding author.

TABLE 1. Certain members belonging to the Appell family.

S.No.	name of the polynomials and related numbers	$A(t)$	series definition	Generating functions
1.	Bernouli polynomials and numbers {10}	$\frac{t}{e^t-1}$	$B_n(x)$ $\sum_{k=o}^{n} B_k x^{n-k}$	$e^{yt}(\frac{t}{e^t-1}) = \sum_{n=o}^{\infty} B_n(x)$ $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$
2.	Euler polynomials and numbers {10}	$\frac{2}{e^t+1}$	$E_n(x) = \sum_{k=0}^{n} \frac{E_k}{2^k}(x-\frac{1}{2})^{n-k}$	$(\frac{2}{e^t+1})e^{yt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$ $\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$
3.	Genocchi polynomials and numbers {16}	$\frac{2t}{e^t+1}$	$G_n(x) = \sum_{k=0}^{n} G_k x^{n-k}$	$(\frac{2t}{e^t+1})e^{yt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}$ $\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}$

Note. It should be noted that in table 1, the Genocchi polynomials $G_n(x)$ do not fulfill all requirements of Appell polynomials, for instance, the degree of $G_n(x)$ is $n-1$, though, the degree of Appell polynomials is n. Therefore, we may put $G_n(x)$ in the class of polynomials sequences which are not considered Appell polynomials in the strong sense [2].

A family of truncated polynomials of noteworthy interest is provided by the series [7, p.597(14)]. These polynomials play an important role in the evaluation of the integral involving product of special functions and also appear in a wide variety of physical problems such as optics and quantum mechanics. The special polynomials of two variables are important from the view point of applications. The extension of two variables truncated exponential polynomials is given by series [7, p. 599(31)].

Truncated exponential polynomial $e_n(x)$ defined by the series

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{1.4}$$

Truncated exponential polynomials define by the generating function

$$\frac{e^x t}{(1-t)} = \sum_{n=0}^{\infty} e_n(x) t^n. \tag{1.5}$$

Generating function of Appell polynomials

$$A(t)e^x t = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1.6}$$

We consider the 2-variable higher order truncated polynomials (2TEP) are defined by the series [8, p. 174(29)]

$$e_n^r(x,y) = n! \sum_{k=0}^{\frac{n}{r}} \frac{y^k x^{n-rk}}{(n-rk)!}, \tag{1.7}$$

and the generating function of (2VTP)

$$\frac{e^x t}{(1-yt^r)} = \sum_{n=0}^{\infty} e_n^r(x,y)\frac{t^n}{n!}. \tag{1.8}$$

1.1. **Truncated exponential based Appell polynomials.** 2-Variable truncated exponential baased Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$ can be defined by the generating function [10]

$$A(t)exp(xt)C_0(-yt^r) = \sum_{n=0}^{\infty} {}_{e^r}A_n(x,y)\frac{t^n}{n!}, \tag{1.9}$$

$$A(t)exp(xt)exp(D_y^{-1}t^r) = \sum_{n=0}^{\infty} {}_{e^r}A_n(x,y)\frac{t^n}{n!}, \tag{1.10}$$

where $C_0(ax)$ is Tricomi function of order zero [3].

Since $C_n(xt) = \sum_{n=0}^{\infty} \frac{(-1^r)(xt)^n}{r^2!}$, and where D_x^{-1} denotes the inverse derivate operator $D_x = \frac{\partial}{\partial x}$ and is defined by $D_x^{-1}\{f(x)\} = \int_0^x f(\xi)d\xi$.

Truncated exponential based Appell polynomials (2VTEAP) can defined in series form as follows

$${}_{e_n^r}A_n(x,y) = n!\sum_{k=0}^{\frac{n}{r}} \frac{A_{n-rk}y^k}{(n-rk)!}.$$

In 2002, He and Ricci exploited the idea to derive the differential equations for the Appell polynomials by the factorization method in [14](see also, [16]). Recently, differential equations for mixed type Appell polynomials are derived by many authors [9, 18, 20]. The applications of recurrence relations, differential equations, and other results of these hybrid special polynomials [4, 11, 12, 13]can be observed in different branches of science as well as the solutions of the new emerging problems. This motivates to establish the differential equations for the truncated exponential Appell polynomials by using the factorization method. We retrace some preliminaries related to the factorization method

Let $\{p_n(x)\}_{k=0}^{\infty}$ be a sequence of polynomials such that $deg(p_n(x)) = n(p_n \in \mathbb{N}_0 := 0, 1, 2, 3, ...)$. These operators satisfying the properties

$$\Phi_n^- p_n(x) = p_{n-1}(x), \tag{1.11}$$

$$\Phi_n^+ p_n(x) = p_{n+1}(x), \tag{1.12}$$

are known as the derivative and multiplicative operators, respectively. Dattoli et. al used the monomiality principle [5, 19] and operational rules in [6] The polynomials sequences $\{p_(x)\}_{n=0}^{\infty}$ satisfying the above equations is then called quasi-monomial.

A property, like as the differential equation.

$$(\Phi_{n+1}^- \Phi_n^+)p_n(x) = p_n(x), \tag{1.13}$$

can be derived using the Φ_n^- and Φ_n^+ operators. For finding the derivative operator Φ_n^- and a multiplicative operator Φ_n^+ as the equation (1.13) holds. Above method used for driving the differential equations via equation(1.13) is known as the factorization method [15].

The article is consolidated as follows. The recurrence relation and the shift operator for the truncated exponential-Appell polynomials are established in section 2. Further, the differential equation, integro-differential equation, and partial differential equation for the family are derived. In section 3, corresponding results for the truncated-Bernoulli polynomials, truncated-Euler polynomials, and truncated-Genocchi polynomials are obtained as an example. The integral equation for the truncated-Appell polynomials and the certain members belonging to this family are derived in the last section.

2. Recurrence relations and differential equation

First, we recall the recurrence relation for the truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$.

Theorem 1. *The truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$ satisfy the recurrence relation:*

$$\begin{aligned}{}_{e_n^r}A_{n+1}(x,y) &= (x+\alpha_0){}_{e^r}A_n(x,y)\\ &+\sum_{k=1}^{n}\binom{n}{k}\alpha_k{}_{e^r}A_{n-k}(x,y)+\frac{n!}{(n-r+1)!}\,rD_y^{-1}{}_{e^r}A_{n-r+1}(x,y),\end{aligned} \tag{2.1}$$

where the coefficient $\{a_k\}_{k\in\mathbb{N}_0}$ are given by the expansion.

$$\frac{A'(t)}{A(t)}=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k. \tag{2.2}$$

Proof. Differentiating both sides of generating function(1.10) of truncated exponential based Appell polynomials (2VTEAP) with respect to t, we have

$$\left(x+rD_y^{-1}t^{r-1}+\frac{A'(t)}{A(t)}\right)A(t)e^{xt+D_y^{-1}t^r}=\sum_{n=0}^{\infty}{}_{e_n^r}A_{n+1}(x,y)\frac{t^n}{n!}. \tag{2.3}$$

Using the equation (2.2) and (1.10) in the above equation and then applying the cauchy product rule in left side of the resultant equation, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{e_n^r}A_{n+1}(x,y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\alpha_k{}_{e_n^r}A_{n-k}(x,y)\right.\\ &\left.+\,x{}_{e_n^r}A_n(x,y)+\frac{n!}{(n-r+1)!}rD_y^{-1}{}_{e_n^r}A_{n-r+1}(x,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Now equating the coefficients of same power of t on the both sides of the above equation and the simplifying the resultant equation,then we get equation (2.1). □

Next, we find shift operators for truncated exponential-Appell polynomials.

Theorem 2. *The Shift operator for the truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$ are given by;*

$$ {}_x\pounds_n^- = \frac{1}{n}D_x, \tag{2.5}$$

$$ {}_y\pounds_n^- = \frac{1}{n}D_y D_x^{-(r-1)}, \tag{2.6}$$

$$ {}_x\pounds_n^+ = (x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k + rD_y^{-1}D_x^{r-1}, \tag{2.7}$$

$$ {}_y\pounds_n^+ = (x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^k D_x^{-k(r-1)} + rD_y^{r-2}D_x^{-(r-1)^2}, \tag{2.8}$$

where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$, *and* $D_x^{-1} = \int_0^x f(\xi)d\xi$.

Proof. Differentiating the generating function of truncated exponential based Appell polynomials (2VTEAP) with respect to x and the equating the coefficient of same power of t from both sides of equation, then we get

$$ \frac{\partial}{\partial x}\left({}_{e_n^r}A_n(x,y)\right) = n{}_{e_n^r}A_{n-1}(x,y), \tag{2.9}$$

so that

$$ \frac{1}{n}\frac{\partial}{\partial x}\left({}_{e_n^r}A_n(x,y)\right) = {}_{e_n^r}A_{n-1}(x,y).$$

Consequently, it follows that.

$$ {}_x\pounds_n^-[{}_{e_n^r}A_n(x,y)] = \frac{1}{n}D_x\{{}_{e_n^r}A_n(x,y)\} = {}_{e_n^r}A_{n-1}(x,y), \tag{2.10}$$

Which proves (2.5)

Next, differentiating the generating function (1.10) with respect to y and then equating the same power of t from the both sides, then we get

$$ \frac{\partial}{\partial y}\{{}_{e_n^r}A_n(x,y)\} = \frac{n!}{(n-r)!}\{{}_{e_n^r}A_{n-r}(x,y)\}, \tag{2.11}$$

from the equation (2.9), can be written as

$$ \frac{\partial}{\partial y}\{{}_{e_n^r}A_n(x,y)\} = n\frac{\partial^{r-1}}{\partial_x^{r-1}} = n{}_{e_n^r}A_{n-1}(x,y). \tag{2.12}$$

Thus,

$$ {}_y\pounds_n^- = \frac{1}{n!}D_y D_x^{-(r-1)}, \tag{2.13}$$

which proves (2.6).

Next, to find the raising operator

$$ {}_{e_n^r}A_{n-k}(x,y) = \left({}_x\pounds_{n-k+1}^- {}_x\pounds_{n-k+2}^- \cdots\cdots {}_x\pounds_n^-\right){}_{e_n^r}A_n(x,y),$$

which on using Eq.(2.10)

$$ {}_{e_n^r}A_{n-k}(x,y) = \frac{(n-k)!}{n!} D_x^k {}_{e^r}A_n(x,y). \tag{2.14} $$

Making use of this equation (2.14) in recurrence relation and in view of the fact that

$$ {}_x\pounds_n^+\{{}_{e_n^r}A_n(x,y)\} = {}_{e_n^r}A_{n+1}(x,y), \text{we find } {}_x\pounds_n^+ = (x+\alpha_0) + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^k + rD_y^{-1} D_x^{(r-1)}, $$

Which proves (2.7).

Now, for ${}_y\pounds^+$ find the raising operator, considered the relation,

$$ {}_{e_n^r}A_{n-k}(x,y) = \left({}_y\pounds_{n-k+1}^- {}_y\pounds_{n-k+2}^- \cdots\cdots {}_y\pounds_{n-k+1}^- \right) {}_{e_n^r}A_n(x,y), $$

which on using equation (2.13)

$$ {}_{e_n^r}A_{n-k}(x,y) = \frac{(n-k)!}{n!} D_y^k D_x^{k(1-r)} \{{}_{e_n^r}A_n(x,y)\}, \tag{2.15} $$

by using of this equation(2.15) in recurrence relation we have

$$ {}_y\pounds_n^+\{{}_{e_n^r}A_n(x,y)\} = {}_{e_n^r}A_{n+1}(x,y), $$

$$ {}_y\pounds_n^+ = (x+\alpha_0) + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^k D_x^{-k(r-2)} + rD_y^{(r-2)} D^{-(r-1)^2}, $$

which proves (2.8) equation. □

Now, we derive differential equation for truncated exponential-Appell polynomials.

Theorem 3. *The differential equation for truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$ satisfy the following equation.*

$$ \left((x+\alpha_0)D_x + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{k+1} + ryD_x^r - n \right) {}_{e_n^r}A_n(x,y) = 0. \tag{2.16} $$

Proof. Considered the factorization method for derivation of differential equation

$$ {}_x\pounds_{n+1}^- {}_x\pounds_n^+\{{}_{e_n^r}A_n(x,y)\} = {}_{e_n^r}A_n(x,y). $$

Putting the value of the shift operators from the equations (2.5) and (2.7)

$$ \left((x+\alpha_0)D_x + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{k+1} + rD_y^{-1} D_x^{(r-1)} D_x - n \right) e_n^r A_n(x,y) = 0, $$

or

$$ \left((x+\alpha_0)D_x + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{k+1} + rD_y^{-1} D_x^{r} - n \right) e_n^r A_n(x,y) = 0, $$

which proves (2.16). □

3. Integral equation

Integral equations occur in many scientific and engineering problems. The integral equations created by the contribution of Mathematical physics models, such as scattering in quantum mechanics, diffraction problems, conformal transformation, and water waves also. Integral equation for the Appell polynomials A(x) and the 2-iterated Appell polynomials and certain members belonging to these families are established in [9]. Now we have derived the integral equation for the truncated exponential based Appell polynomials.

Theorem 4. *For the truncated exponential based Appell polynomials (TEAP)* ${}_{e_n^r}A_n(x,y)$*, satisfy the following homogeneous volterra integral equation*

$$\begin{aligned}\phi(x) = &-\frac{\alpha_1}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!}+\mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!}+......+\mathbb{P}_2(x)+\mathbb{P}_1\right)\\ &-\frac{\alpha_0}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!}+\mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!}+\mathbb{P}_2\frac{x^2}{2!}+\mathbb{P}x+n\mathbb{R}_{n-1}\right)\\ &-\frac{1}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!}+\mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!}+......+\mathbb{P}_2\frac{x^3}{2!}+\mathbb{P}x^2+n\mathbb{R}n-1x\right)\\ &+\frac{n}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!}+\mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!}+........+\mathbb{P}_1\frac{x^2}{2!}+n\mathbb{R}_{n-1}x+\mathbb{R}_n\right)\\ &-\frac{1}{ry}\int_0^x\left(\alpha_1\frac{(x-\xi)^{r-3}}{(r-3)!}+(x+\alpha_0)\frac{(x-\xi)^{r-2}}{(r-2)!}-n\frac{(x-\xi)^{r-1}}{(r-1)!}\right)\phi(\xi)d\xi. \quad (3.1)\end{aligned}$$

Proof. consider the differential equation of truncated exponential-Appell polynomial for $k=1$ in the following form

$$\left(D_x^r+\{(x+\alpha_0)D_x+\alpha_1 D_x^2-n\}\frac{1}{ry}\right){}_{e_n^r}A_n(x,y)=0.$$

In the generating function (1.10), with $y=0$,

$$A(t)exp(xt)=\sum_{n=0}^{\infty}{}_{e_n^r}A_n(x,0)\frac{t^n}{n!}=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!}$$

using the $A(t)=\sum_{n=0}^{\infty}A_n\frac{t^n}{n!}$ and expanding the exponential in left hand side and then apply cauchy product rule,

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}A_{n-k}x^k.$$

The initial condition obtained

$${}_{e_n^r}A_n(x,0)=A_n(x)=\sum_{k=0}^{n}\binom{n}{k}A_{n-k}x^k,$$

letting $A_n(x) = \mathbb{R}_n$ Then the derivative of $A_n(x)$ with respect to x,

$$\frac{d}{dx}{}_{e_n^r}A_n(x,0) = nA_{n-1}(x) = n\sum_{k=0}^{n}\binom{n-1}{k}A_{n-1-k}x^k = n\mathbb{R}_{n-1},$$

$$\frac{d^2}{dx^2}{}_{e_n^r}A_n(x,0) = n(n-1)A_{n-2}(x) = \prod_{k=0}^{1}(n-k)\mathbb{R}_{n-2} = \mathbb{P}_1,$$

Since $\mathbb{R}_n = {}_{e_n^r}A_n(x) = A_n(x)$;

$$D_x^{r-2} = \{{}_{e^r}A_n(x,0)\} = n(n-1)(n-2)......(n-r+3)\mathbb{R}_{n-r+2} = \mathbb{P}_{r-3} = \prod_{k=0}^{r-3}(n-k)\mathbb{R}_{n-r+2},$$

$$D_x^{r-1}\{{}_{e_n^r}A_n(x,0)\} = n(n-1)(n-2)........(n-r+1)\mathbb{R}_{n-r+1} = \mathbb{P}_{r-2} = \prod_{k=0}^{r-2}(n-k)\mathbb{R}_{n-r+1}. \quad (3.2)$$

Consider the equation,

$$D_x^r\{{}_{e_n^r}A_n(x,y)\} = \phi(x). \quad (3.3)$$

Now, integrating the equation(3.3) and using the initial conditions, we have

$$D_x^{r-1}\{{}_{e_n^r}A_n(x,y)\} = \int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2},$$
$$D_x^{r-2}\{{}_{e_n^r}A_n(x,y)\} = \int_0^x \phi(\xi)d^2\xi + \mathbb{P}_{r-3},$$

$$D_x^2\{{}_{e_n^r}A_n(x,y)\} = \int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1,$$

$$D_x\{{}_{e_n^r}A_n(x,y)\} = \int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + ... + \mathbb{P}_1 x + ... + n\mathbb{R}n-1,$$

$${}_{e_n^r}A_n(x,y) = \int_0^x \phi(\xi)d\xi^r + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n, \quad (3.4)$$

where

$$\mathbb{P}_{r-s} = \prod_{k=0}^{r-s}(n-k)\mathbb{R}_{n-r+(s-1)}, s = r-1, r-2, ...3, 2.$$

Using the expression (3.4) in equation(3.1), we find

$$\phi(x) = -\frac{(x+\alpha_0)}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}_1\frac{x}{1!} + n\mathbb{R}_{n-1}\right)$$
$$-\frac{\alpha_1}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1\right)$$
$$+\frac{n}{ry}\left(\int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right),$$

which proves assertion (3.1). □

Corollary 1. *For the truncated exponential based Bernoulli polynomials the following homogeneous volterra integral equation putting* $\alpha_0 = -\frac{1}{2}, \alpha_1 = -\frac{1}{12}$ *in the integral equation* (3.1)

$$\phi(x) = -\frac{(x-\frac{1}{2})}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}_1\frac{x}{1!} + n\mathbb{R}_{n-1}\right)$$
$$-\frac{-\frac{1}{12}}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1\right)$$
$$+\frac{n}{ry}\left(\int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right).$$

Corollary 2. *For the truncated exponential based Euler polynomials the following homogeneous integral equation putting* $\alpha_0 = -\frac{1}{2}, \alpha_1 = -\frac{1}{4}$ *in the integral equation* (3.1)

$$\phi(x) = -\frac{(x-\frac{1}{2})}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}_1\frac{x}{1!} + n\mathbb{R}_{n-1}\right)$$
$$-\frac{-\frac{1}{4}}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1\right)$$
$$+\frac{n}{ry}\left(\int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right).$$

Corollary 3. *For the truncated exponential based Genocchi polynomials the following homogeneous integral equation putting* $\alpha_0 = 1, \alpha_1 = -1$ *in the equation* (3.1)

$$\phi(x) = -\frac{(x+1)}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}_1\frac{x}{1!} + n\mathbb{R}_{n-1}\right)$$
$$-\frac{-1}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1\right)$$
$$+\frac{n}{ry}\left(\int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right).$$

4. Integro-differential equation for truncated exponential based Appell polynomials (2VTEAP)

Theorem 5. *The integro differential equation of truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$ satisfy following equation*

$$\left((x+\alpha_0)D_x + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^k D_x D_x^{-k(r-1)} + rD_y^{(r-2)} D_x^{-(r-1)^2+1} - nD_x\right) {}_{e_n^r}A_n(x,y) = 0. \tag{4.1}$$

Proof. By the factorization method,

using the pair of shift operators ${}_x\pounds_{n+1}^{-}$ and ${}_y\pounds_n^{+}$, we have assertion (4.1) proved.

$${}_x\pounds_{n+1}^{-} {}_y\pounds_n^{+}\{{}_{e_n^r}A_n(x,y)\} = {}_{e_n^r}A_n(x,y),$$

putting the value of shift operator from the equation (2.5) and (2.8) in the left hand side above equation then, we get equation (4.1). □

Corollary 4. *By differentiating equation of integro-differential equation n-times with respect to y then, we get a partial differential equation of truncated exponential based Appell polynomials (2VTEAP) ${}_{e_n^r}A_n(x,y)$*

$$\left((x+\alpha_0)D_x D_y^n + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y^{n+k} D_x^{-k(r-1)+1} + rD_y^{(r-2)+n} D_x^{(r-1)^2+1} - nD_x D_y^n\right) e_n^r A_n(x,y) = 0. \tag{4.2}$$

We derived the recurrence relation, differential equation, integro differential equation, and partial differential equation for the special case of Appell polynomials by considering the following examples.

5. Examples

We solve the following examples.

Examples 3.1 : Taking $A(t) = \frac{t}{(e^t-1)}$, that is when Appell polynomials reduced to the Bernoulli polynomials (2VTEBP) and using the generating function

$$\frac{A'(t)}{A(t)} = -\sum_{n=0}^{\infty} \frac{B_{n+1}(1)}{n+1} \frac{t^n}{n!}.$$

$$\text{by} \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!} = -\sum_{n=0}^{\infty} \frac{B_{n+1}(1)}{n+1} \frac{t^n}{n!}.$$

consequently,

$$\alpha_n = -\frac{B_{n+1}(1)}{n+1}, \alpha_0 = -\frac{1}{2}.$$

Example 3.2 : Taking $\frac{A'(t)}{A(t)} = \left(\frac{2}{e^t+1}\right)$, (that is, when the reduced to the Euler polynomials (2VTEBP) and using the generating function from table 1)

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \frac{E_n}{2}\frac{t^n}{n!}.$$

$$\text{since} \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{E_n}{2}\frac{t^n}{n!}.$$

$$\alpha_n = \frac{E_n}{2}, \alpha_0 = -\frac{1}{2}.$$

Example 3.3 : Taking $A(t) = \left(\frac{2t}{e^t+1}\right)$ (that is, when the (TEAP) reduced to the Gennochi polynomials (2VTEGP) and using the generating function from table 1.)

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \frac{G_n}{2}\frac{t^n}{n!}.$$

Using equation (2.2) in the above equation, we find

$$\sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{G_n}{2}\frac{t^n}{n!}.$$

consequently,

$$\alpha_n = \frac{G_n}{2}, \alpha_0 = 1, \alpha_1 = -1.$$

Table 2. Results for TEBP, TEEP and TEGP

1.	Recurrence relation ${}_{e_n^r}B_{n+1}(x,y) = (x-\frac{1}{2}){}_{e_n^r}B_n(x,y) - \sum_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}}{k+1}{}_{e_n^r}B_{n-k}(x,y) + \frac{n!}{(n-r+1)!}rD_y^{-1}{}_{e_n^r}B_{n-r+1}(x,y)$
2.	shift operators ${}_x\pounds_n^- = \frac{1}{n}D_x$ ${}_y\pounds_n^- = \frac{1}{n}D_yD_x^{-(r-1)}$ ${}_x\pounds_n^+ = (x-\frac{1}{2}) - \sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_x^k + rD_y^{-1}D_x^{(r-1)}$ ${}_y\pounds_n^+ = (x-\frac{1}{2}) - \sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^kD_x^{-k(r-1)} + rD^{(r-2)}D_x^{-(r-1)^2}$
3.	Differential equation $[(x-\frac{1}{2})D_x - \sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_x^{k+1} + ryD_x^r - n]{}_{e_n^r}B_n(x,y) = 0$
4.	Integro differential equation $[(x-\frac{1}{2})D_x - \sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^kD_x^{-k(r-1)+1} + rD_y^{(r-2)}D_x^{-(r-1)^2+1} - nD_x]{}_{e_n^r}B_n(x,y) = 0$
5.	Partial differential equation $\{(x-\frac{1}{2})D_xD_y^n - \sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^{n+k}D_x^{-k(r-1)+1} + ryD_y^{(r-2)+n}D_x^{-(r-1)^2+1} - nD_xD_y^n\}{}_{e_n^r}B_n(x,y) = 0$
1.	Recurrence relation ${}_{e_n^r}E_{n+1}(x,y) = (x-\frac{1}{2}){}_{e_n^r}E_n(x,y) + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}E_k{}_{e_n^r}E_{n-k}(x,y) + \frac{n!}{(n-r+1)!}rD_y^{-1}{}_{e_n^r}E_{n-r+1}(x,y)$
2.	shift operators ${}_x\pounds_n^- = \frac{1}{n}D_x$ ${}_y\pounds_n^- = \frac{1}{n}D_yD_x^{-(r-1)}$ ${}_x\pounds_n^+ = (x-\frac{1}{2}) + \frac{1}{2}\sum_{k=1}^{n}\frac{E_k}{(k)!}D_x^k + rD_y^{-1}D_x^{(r-1)}$ ${}_y\pounds_n^+ = (x-\frac{1}{2}) + \frac{1}{2}\sum_{k=1}^{n}\frac{E_k}{(k)!}D_y^kD_x^{-k(r-1)} + rD^{(r-2)}D_x^{-(r-1)^2}$
3.	Differential equation $[(x-\frac{1}{2})D_x + \frac{1}{2}\sum_{k=1}^{n}\frac{E_k}{(k)!}D_x^{k+1} + ryD_x^r - n]{}_{e_n^r}E_n(x,y) = 0$
4.	Integro differential equation $[(x-\frac{1}{2})D_x + \frac{1}{2}\sum_{k=1}^{n}\frac{E_k}{(k)!}D_y^kD_x^{-k(r-1)+1} + rD_y^{(r-2)}D_x^{-(r-1)^2+1} - nD_x]{}_{e_n^r}E_n(x,y) = 0$
5.	Partial differential equation $\{(x-\frac{1}{2})D_xD_y^n + \frac{1}{2}\sum_{k=1}^{n}\frac{E_k}{(k)!}D_y^{n+k}D_x^{-k(r-1)+1} + ryD_y^{(r-2)+n}D_x^{-(r-1)^2+1} - nD_xD_y^n\}{}_{e_n^r}E_n(x,y) = 0$
1.	Recurrence relation ${}_{e_n^r}G_{n+1}(x,y) = (x+1){}_{e_n^r}G_n(x,y) + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}G_k{}_{e_n^r}G_{n-k}(x,y) + \frac{n!}{(n-r+1)!}rD_y^{-1}{}_{e_n^r}G_{n-r+1}(x,y)$
2.	Shift operators ${}_x\pounds_n^- = \frac{1}{n}D_x$ ${}_y\pounds_n^- = \frac{1}{n}D_yD_x^{-(r-1)}$ ${}_x\pounds_n^+ = (x+1) + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{(k)!}D_x^k + rD_y^{-1}D_x^{(r-1)}$ ${}_y\pounds_n^+ = (x+1) + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{(k)!}D_y^kD_x^{-k(r-1)} + rD^{(r-2)}D_x^{-(r-1)^2}$
3.	Differential equation $[(x+1)D_x + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{(k)!}D_x^{k+1} + ryD_x^r - n]{}_{e_n^r}G_n(x,y) = 0$
4.	Integro differential equation $[(x+1)D_x + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{(k)!}D_y^kD_x^{-k(r-1)+1} + rD_y^{(r-2)}D_x^{-(r-1)^2+1} - nD_x]{}_{e_n^r}G_n(x,y) = 0$
5.	Partial differential equation $\{(x+1)D_xD_y^n + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{(k)!}D_y^{n+k}D_x^{-k(r-1)+1} + ryD_y^{(r-2)+n}D_x^{-(r-1)^2+1} - nD_xD_y^n\}{}_{e_n^r}G_n(x,y) = 0$

6. Acknowledgments

This work was supported by the Dongguk University research fund of 2020.

References

[1] P. Appell, Sur une classe de polynômes, *Ann. Sci. Éc. Norm. Supér Appl.*, **9(2)** (1880), 119-144.

[2] L. Aceto, H.R. Maionek and G. Tomaz; A unified matrix approach to the representation of Appell polynomials, *Integral Transf.Spec.Funct.*, **26(6)** (2015), 426-441.

[3] L.C. Andrews; *Special functions for engineers and applied mathematicians*, Macmillan, New York, (1985).

[4] R. Das and A. Sannigrahi; The factorization method and its applications in quantum chemistry, *J. Chem. Educ.*, **58** (1981), 383-390.

[5] G. Dattoli; Hermite–Bessel and Laguerre–Bessel functions: A by-product of the monomiality principle, in: *Advanced Special Functions and Applications.* Melfi(1999), Proc. Melfi Sch. Adv. Top. Math. Phys. 1, Aracne, Rome (2000), 147-164.

[6] G. Dattoli, C. Cesarano and D. Sacchetti; A note on the monomiality principle and generalized polynomials, *J. Math. Anal. Appl.*, **227** (1997), 98-111.

[7] G. Dattoli, C. Cesarano, and D. Sacchetti; A note on truncated polynomials, *Appl. Math. Comp.*, **134(1-2)** (2003), 595-605.

[8] G. Dattoli, M. Migliorati and H.M. Srivastava; A class of Bessel summation formulas and associated operational method, *Frac. Calcu. Appl. Anal*, **7(2)**, (2004), 169-176.

[9] S. Khan and M. Riyasat; Differential and integral equation for 2-iterated Appell polynomials, *J. Comput. Appl. Math.* **306** (2016), 116-132.

[10] S. Khan, G. Yasmin and N. Ahmad; A note on truncated Exponential-based Appell polynomials, *Bull. Malays. Math. Sci. Soc.*, **26(6)** (2014), 373-388.

[11] D.S. Kim and T. Kim; Degenerate Sheffer sequences and λ-Sheffer sequences, *J. Math. Anal. Appl.*, **493(1)** (2021), 124521, 21 pp.

[12] T. Kim, D.S. Kim, H. Lee and C.-Jang; A note on degenerate derangement polynomials and numbers, *AIMS Math.*, **6(6)** (2021), 6469–6481.

[13] T. Kim and D.S. Kim; On λ-Bell polynomials associated with umbral calculus, *Russ. J. Math. Phys.*, **24(1)** (2017), 69–78.

[14] M.X. He and P.E. Ricci; Differential equation of the Appell polynomials via factorization method, *J. Comput. Appl. Math.*, **139(2)** (2002), 231-237.

[15] L. Infeld and T.E. Hull; The factorization method, *Rev. Modern Phys.*, **23** (1951), 21-68.

[16] M.A. Ozarslan and B. Yilmaz; A set of finite order differential equations for the Appell polynomials, *J. Comput. Appl. Math.* **259A** (2014), 108-116.

[17] S. Roman, *The umbral calculus*, Springer, New York, 2005.

[18] H.M. Srivastava, M.A. Ozarslan and B.Yilmaz; Some families of differential equation associated with the hermite based Appell polynomials and other classes of the hermite-based polynomials, *Filomat.*, **28(4)** (2014), 695–708

[19] J.F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.*, **73** (1941), 333-366.

[20] B. Yılmaz and M.A. Özarslan, Differential equations for the extended 2D Bernoulli and Euler polynomials, *Adv. Differ. Equ.*, **107** (2013), Article ID 107

Raziya Sabri: Department of Natural and Applied Sciences, School of Science and Technology, Glocal University, Saharanpur 247121, India.
Email address: razisabri786@gmail.com

Mohammad Shadab: Department of Natural and Applied Sciences, School of Science and Technology, Glocal University, Saharanpur 247121, India.
Email address: shadabmohd786@gmail.com

Taekyun Kim: Department of Mathematics, Kwangwoon University, Seoul 139701121, Republic of Korea.
Email address: tkkim@kw.ac.kr

Jose Luis Lopez-Bonilla: Department of Electrical and Electronic Engineering, National Polytechnic Institute, Mexico.
Email address: joseluis.lopezbonilla@gmail.com

Byung Moon Kim: Department of Mechanical System Engineering, Dongguk University, 123 Dongdae-ro, Gyungju-si, Gyeongsangbuk-do, 38066, Republic of Korea.
Email address: kbm713@dongguk.ac.kr